Thermal Instability in Rivlin-Ericksen Elastico-Viscous Fluids in Hydromagnetics

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The thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform vertical magnetic field is considered. For stationary convection, a Rivlin-Ericksen elastico-viscous fluid behaves like a Newtonian fluid. The magnetic field has a stabilizing effect. It is found that the presence of a magnetic field introduces oscillatory modes which were non-existent in its absence. The sufficient condition for the non-existence of overstability is also obtained.

Introduction

A detailed account of thermal instability of electrically conducting fluids in the presence of a magnetic field has been given by Chandrasekhar [1]. There it is shown that a uniform magnetic field inhibits the onset of thermal convection. The thermal instability of a Maxwell fluid in hydromagnetics have been studied by Bhatia and Steiner [2]. They have found that the magnetic field stabilizes a viscoelastic (Maxwell) fluid just as Newtonian fluids. In [3] the stability of a layer of an electrically conducting Oldroyd fluid [4], heated from below in the presence of a magnetic field, has been studied. It was found that the magnetic field has a stabilizing influence.

There are many elastico-viscous fluids which cannot be characterized by Maxwell's or Oldroyd's constitutive relations [4], e.g. the Rivlin-Ericksen fluid. Sharma and Kumar [5] have studied the effect of rotation on the thermal instability of that fluid.

In the present paper we study the thermal instability of the Rivlin-Ericksen fluid in the presence of a uniform magnetic field.

2. Description of the Problem and Dispersion Relation

We consider an infinite horizontal fluid layer of depth d, which is acted on by a uniform vertical magnetic field $\boldsymbol{H}(0, 0, H)$ and the force of gravity $\boldsymbol{g}(0, 0, -g)$. This layer is heated from below such that a steady downward temperature gradient $\beta(=|\mathrm{d}T/\mathrm{d}z|)$ is maintained.

The initial state is one in which the velocity, density, pressure and temperature at any point in the fluid are respectively given by

$$\boldsymbol{q} = \mathrm{o}, \quad \varrho = \varrho(z), \quad p = p(z) \quad \text{and} \quad T = T(z).$$

Let $\boldsymbol{q}(u, v, w)$, $\boldsymbol{h}(h_x, h_y, h_z)$, $\delta\varrho$, δp and θ denote the perturbations in velocity (0, 0, 0), magnetic field $\boldsymbol{H}$, density ϱ, pressure p and temperature T, respectively. Then the linearized hydromagnetic perturbation equations (Chandrasekhar [1], Sharma and Kumar [5]) are

$$\frac{\partial \boldsymbol{q}}{\partial t} = -\frac{1}{\varrho_0}\nabla\delta p + \left(v + v'\frac{\partial}{\partial t}\right)\nabla^2\boldsymbol{q} + \boldsymbol{g}\frac{\delta\varrho}{\varrho_0} + \frac{\mu_e}{4\pi\varrho_0}(\nabla\times\boldsymbol{h})\times\boldsymbol{H}, \tag{1}$$

$$\nabla\cdot\boldsymbol{q} = 0, \tag{2}$$

$$\frac{\partial\theta}{\partial t} + (\boldsymbol{q}\cdot\nabla)T = \kappa\nabla^2\theta, \tag{3}$$

$$\frac{\partial\boldsymbol{h}}{\partial t} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{q} + \eta\nabla^2\boldsymbol{h}, \tag{4}$$

$$\nabla\cdot\boldsymbol{h} = 0. \tag{5}$$

Here we follow the analysis and notations as in [5], analyzing the disturbances into normal modes and assuming that the perturbation quantities are of the form

$$[w, \theta, h_z] = [W(z), \Theta(z), K(z)] \cdot \exp(i k_x x + i k_y y + n t). \tag{6}$$

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Equations (1)–(5), using (6), give

$$\left[\sigma(D^2-a^2)W+\left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta-\frac{\mu_e H d}{4\pi\varrho_0\nu}(D^2-a^2)DK\right] = (1+F\sigma)(D^2-a^2)^2 W, \quad (7)$$

$$(D^2-a^2-p_2\sigma)K=-\left(\frac{Hd}{\eta}\right)DW, \quad (8)$$

$$(D^2-a^2-p_1\sigma)\Theta=-\left(\frac{\beta d^2}{\kappa}\right)W. \quad (9)$$

Following Sharma and Kumar [5], and assuming the case of two free boundaries, we obtain the dispersion relation

$$R_1=\frac{(1+x)(1+x+ip_1\sigma_1)\cdot[(1+x+ip_2\sigma_1)\{i\sigma_1+(1+Fi\sigma_1\pi^2)(1+x)\}+Q_1]}{x(1+x+ip_2\sigma_1)}, \quad (10)$$

where

$$R_1=\frac{g\alpha\beta d^4}{\nu\kappa\pi^4},\quad Q_1=\frac{\mu_e H^2 d^2}{4\pi\varrho_0\nu\eta\pi^2},\quad a^2=\pi^2 x,\quad \frac{\sigma}{\pi^2}=i\sigma_1. \quad (11)$$

σ can be complex. Here we consider the overstable mode, and so σ_1 is real in (10).

For the case of stationary convection, i.e. $\sigma=0$, (10) reduces to

$$R_1=\left(\frac{1+x}{x}\right)[(1+x)^2+Q_1], \quad (12)$$

the result obtained by Chandrasekhar [1].

We thus find that for stationary convection the viscoelasticity parameter F vanishes with σ and the Rivlin-Ericksen elastico-viscous fluid behaves like Newtonian fluid.

To study the effect of magnetic field, we examine the nature of $\frac{dR_1}{dQ_1}$ obtained from (10). Equating its real and imaginary parts, we obtain

$$\frac{dR_1}{dQ_1}=\frac{1+x}{x}, \quad (13)$$

which is always positive. The magnetic field thus has a stabilizing influence.

3. Stability of the System and Oscillatory Modes

Multiplying (7) by W^*, the complex conjugate of W, integrating over the range of z and making use of (8) and (9) together with the boundary conditions $W=D^2W=0$, $\Theta=0$ at $z=0, 1$; we obtain

$$-\sigma I_1+\frac{g\alpha\kappa a^2}{\nu\beta}(I_2+p_1\sigma^* I_3)+\frac{\mu_e\eta}{4\pi\varrho_0\nu}(I_4-p_2\sigma^* I_5)=(1+F\sigma)I_6, \quad (14)$$

where

$$\begin{aligned}
I_1&=\int_0^1(|DW|^2+a^2|W|^2)\,dz,\\
I_2&=\int_0^1(|D\Theta|^2+a^2|\Theta|^2)\,dz,\\
I_3&=\int_0^1(|\Theta|^2)\,dz,\\
I_4&=\int_0^1\{|(D^2-a^2)K|^2\}\,dz,\\
I_5&=\int_0^1(|DK|^2+a^2|K|^2)\,dz,\\
I_6&=\int_0^1(|D^2W|^2+2a^2|DW|^2+a^4|W|^2)\,dz,
\end{aligned} \quad (15)$$

and σ^* is the complex conjugate of σ. The integrals I_1–I_6 are all positive definite. Putting $\sigma=\sigma_r+i\sigma_i$ in (14) and equating the real and imaginary parts, we obtain

$$\sigma_r\left[-I_1+\frac{g\alpha\kappa a^2}{\nu\beta}p_1 I_3-\frac{\mu_e\eta}{4\pi\varrho_0\nu}p_2 I_5-FI_6\right]=-\frac{g\alpha\kappa a^2}{\nu\beta}I_2-\frac{\mu_e\eta}{4\pi\varrho_0\nu}I_4+I_6 \quad (16)$$

and

$$\sigma_i\left[-I_1-\frac{g\alpha\kappa a^2}{\nu\beta}p_1 I_3+\frac{\mu_e\eta}{4\pi\varrho_0\nu}p_2 I_5-FI_6\right]=0. \quad (17)$$

It is evident from (16) that σ_r is positive or negative. The system is therefore stable or unstable. From (17) it follows that σ_i may be zero or non-zero, meaning that the modes may be non-oscillatory or oscillatory. The oscillatory modes are introduced by the magnetic field.

4. The Case of Overstability

Here we discuss if instability may occur as an overstability. When the marginal state is oscillatory, we must have $\sigma_r=0$, $\sigma_i\neq 0$.

Since for overstability we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (10) admits solutions with σ_1 real.

Separating the real and imaginary parts of (10) and eliminating R_1 between the resulting equations, we obtain

$$[p_2^2(1+\alpha F\pi^2+p_1)]\sigma_1^2 + [\alpha^2(1+\alpha F\pi^2+p_1)+Q_1(p_1-p_2)]=0\,, \quad (18)$$

where $1+x=\alpha$.

Since σ_1 is real for overstability, the value of σ_1^2 is positive. Equation (18) shows that this is impossible if $p_1 > p_2$. Therefore, the sufficient condition for the non-existence of overstability is

$$p_1 > p_2\,, \quad (19)$$

which implies that

$$\kappa < \eta\,. \quad (20)$$

The condition $\kappa < \eta$ is thus a sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

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